# A GENERALIZATION OF A PROPOSITION BY LIAPUNOV ON THE EXISTENCE OF PERIODIC SOLUTIONS

## (OBOBSHCHENIE ODNOGO PREDLOZHENIIA LIAPUNOVA O SUSHCHESTROVANII PERIODICHESKIKH RESHENII)

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Let us consider the system of equations

$$\frac{dx}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n + X_s(x_1, \ldots x_n) \qquad (s=1,\ldots,n) \tag{1}$$

where $a_{si}$ are constants, $X_s$ analytic functions of $x_1, \ldots, x_n$. In the neighborhood of the point $x_s = 0$, the series expansion of these functions in powers of $x_1, \ldots, x_n$ begins with terms not below the second order, $X_s(0, \ldots, 0)$.

Liapunov has shown that if among the roots of the equation

$$|a_{si} - \delta_{si}\lambda| = 0 \tag{2}$$

there exists a pair of pure imaginaries of the form $\pm\lambda i$, and if the remaining roots have negative real parts, and if the system (1) has a holomorphic integral, independent of $t$, of the form

$$M(x_1, \ldots, x_n) + \Phi(x_1, \ldots, x_n) = C \tag{3}$$

where $M$ is a quadratic form, and where $x_1, \ldots, x_n$ is the first integral corresponding to the root $\lambda i$ of the linear system of differential equations of the first approximation

$$\frac{dx_s}{dt} = a_{s1}x_1 + \ldots + a_{sn}x_n \tag{4}$$

and if the function $\Phi(x_1, \ldots, x_n)$ has an expansion in $x_1, \ldots, x_n$ beginning with terms of at least the third order, then the system of equation (1) has a family of periodic solutions, depending on one real parameter. The period of this periodic solution is also dependent on one parameter, and will be a holomorphic function of that parameter. The solution $x = 0$ is stable in the Liapunov sense, and any solution with initial conditions sufficiently near $x_s = 0$, will, as $t \to \infty$, approach one of the periodic solutions of the said family.

We will note that this proposition is usually formulated on the assumption that the roots $\pm \lambda i$ are excluded from the system of equations of the first approximation (4).

The Liapunov proposition cited can be generalized.

1. We will assume that equation (2) has $m$ critical roots with simple elementary divisors of the form $\pm N\lambda i$, where $N$ is a positive integer or zero, and that among these roots there exists at least one pair of roots of the form $\pm \lambda i$. The remaining roots of equation (2) have negative parts.

2. We will further assume that the system of equations (1) has $m - 1$ holomorphic integrals of the form

$$M_k(x_1, \ldots, x_n) + \Phi_k(x_1, \ldots, x_n) = C_k \qquad (k=1,\ldots,m-1) \tag{5}$$

where $M_k(x_1, \ldots, x_n)$ are first integrals of system (4), corresponding to critical roots of the form $\pm \lambda iN$, and themselves representing forms with constant coefficients of the first or second order. The integrals $M_k$ are independent among themselves. The functions $\Phi_k$ are expanded in series in the neighborhood of the point $x_s = 0$, starting with terms whose order is higher by one than the order of the corresponding integral $M_k$.

Let $(\phi_{1k}, \ldots, \phi_{nk})$ $(k = 1, \ldots, n)$ be a periodic solution of (4), corresponding to the critical roots. It is clear that the expression

$$\left(\frac{\partial M_k}{\partial x_i}\right)_{xj} \qquad \binom{i=1,\ldots,m}{j=1,\ldots,n}$$

where the index $x_j$ indicates the substitution

$$x_j = \beta_1\varphi_{j1} + \ldots + \beta_m\varphi_{jm} \qquad (\beta_i = \text{const}) \qquad (j=1,\ldots n)$$

is a periodic solution of a linear system, the conjugate of (4).

Thus the product

$$\left\|\begin{matrix} \dfrac{\partial M_1}{\partial x_1} & \ldots & \dfrac{\partial M_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \dfrac{\partial M_{m-1}}{\partial x_1} & \ldots & \dfrac{\partial M_{m-1}}{\partial x_n} \\ 0 & \ldots & 0 \end{matrix}\right\|_{x_j} \times \left\|\begin{matrix} \varphi_{11}\varphi_{12} & \ldots & \varphi_{1m} \\ \varphi_{21}\varphi_{22} & \ldots & \varphi_{2m} \\ \ldots & \ldots & \ldots \\ \varphi_{n1}\varphi_{n2} & \ldots & \varphi_{nm} \end{matrix}\right\| = B \tag{6}$$

$$x_j = \beta_1\varphi_{j1} + \ldots + \beta_m\varphi_{jm} \qquad (j=1,\ldots n)$$

equals the constant matrix

$$B = \|b_{ij}\| \qquad \binom{i=1,\ldots m-1}{j=1,\ldots m}$$

It can easily be shown that on the assumptions made about critical roots (to which correspond simple elementary divisors), the rank of the matrix $B$ is $m - 1$. In fact it is clear that no row of the matrix can consist of zeros, for otherwise the system (4) would have a solution corresponding to one of the critical roots with a secular term, since $b_{i1} = b_{i2} = \dots b_{im} = 0$ is the condition for the existence of a periodic solution for the system of linear nonhomogeneous equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n + \varphi_{si}(t) \qquad (s = 1, \dots, n)$$

This last contradicts the assumption that simple elementary divisors correspond to the critical roots. Let us now assume that the rank of $B$ is less than $m - 1$. We can then choose constants $l_1, \dots, l_m$ such that the first row in matrix $B$ will become zero, e.g. if instead of the first periodic solution $(\phi_{11}, \dots, \phi_{n1})$ we take the combinations of periodic solutions

$$\sum_{\sigma=1}^{m} l_\sigma \varphi_{1\sigma}, \dots, \sum_{\sigma=1}^{m} l_\sigma \varphi_{n\sigma}$$

Hence it follows that the rank of matrix $B$ equals $m - 1$.

*Theorem 1.* If assumptions 1 and 2 are fulfilled, then the system of equations (1) has a family of periodic solutions depending on $m - 1$ real parameters. The period of this solution will be a holomorphic function of these parameters.

The solution $x_s = 0$ is stable in the Liapunov sense, and any solution with initial conditions sufficiently near $x_s = 0$ will tend, as $t \to \infty$, to one of the solutions of the family.

*Remark:* The validity of this assumption for the case when, in addition to the critical roots $\pm \lambda i$, equation (2) also has zero roots to which correspond linear forms, was given by Liapunov in a note to the proof of his theorem [1] (p. 253).

We will sketch the idea of a proof. Let us take the system of integro-differential equations

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n + X_s(x_1, \dots, x_n)(1+\tau) + (a_{s1}x_1 + \dots + a_{sn}x_n)\tau + \sum_{i=1}^{m} w_i \varphi_{si} \qquad (7)$$

where

$$w_i = -\frac{\lambda}{2\pi} \int_0^{2\pi/\lambda} \left[ \sum_{j=1}^{n} X_1(x_1, \dots, x_n)(1+\tau) + (a_{s1}x_1 + \dots + a_{sn}x_n)\tau \right] \psi_{ji}\, dt \qquad (8)$$

and where the functions $\psi_{1i}, \dots, \psi_{ni}$ are periodic solutions of the conjugate system to system (4) of period $2\pi/\lambda$.

This system has a periodic solution depending on $m - 1$ parameters $\beta_1, \dots, \beta_{m-1}$ and the parameter $\tau$ of the form

$$x_s = \varphi_{s1}\beta_1 + \dots + \varphi_{sm-1}\beta_{m-1} + \Phi_s(t, \beta_1, \dots, \beta_{m-1}, \tau) \qquad (9)$$

where $\Phi_s$ are periodic functions of time and analytic functions of $\beta$ and $\tau$, whose series expansion begins with terms of the second order in the neighborhood of the point $\beta = \tau = 0$.

The functions $w_i$ will thus have the form

$$w_i(\beta_1, \ldots, \beta_{m-1}, \tau) \equiv -\frac{\lambda}{2\pi}[(D_{i1}\beta_1 + \ldots + D_{im-1}\beta_{m-1})\tau + (1+\tau)P_i(\beta_1, \ldots, \beta_{m-1}, \tau)] \qquad (i=1,\ldots,m) \tag{10}$$

For the system (1) to have a periodic solution of period $(2\pi/\lambda)(1+\tau)$, it is necessary and sufficient for the system of equations

$$w_i(\beta_i, \ldots, \beta_{m-\lambda}, \tau) = 0 \qquad (i=1,\ldots,m) \tag{11}$$

to have a solution in the neighborhood of the point $\beta = \tau = 0$. The propositions formulated are an immediate consequence of the results published in [2].

Let us now assume that the integrals (5) hold good. We will substitute in them the periodic solution (9) of the auxiliary system (7), (8). We get

$$M_k(x_1(t, \beta, \tau), \ldots) + \Phi_k(x_1(t, \beta, \tau), \ldots) \equiv C_k(t, \beta_1, \ldots \beta_{m-1}, \tau) \qquad (k=1,\ldots,m-1)$$

Differentiating these identities with respect to $t$, and taking into consideration that $x_1(t, \beta, \tau), \ldots, x_n(t, \beta, \tau)$ is a periodic solution of the auxiliary svstem, and that $C_k(t, \beta_1, \ldots, \beta_{m-1}, \tau)$ is a periodic function of $t$, of period $2\pi/\lambda$, we will next integrate these identities with respect to $t$ between the limits 0 and $2\pi/\lambda$. We thus get the system of linear homogeneous equations

$$[b_{i1} + (\ldots)]\,w_1(\beta_1, \ldots, (\beta_{m-1}, \tau) + \ldots + [b_{jm} + (\ldots)]\,w_m(\beta_1, \ldots, \beta_{m-1}, \tau) = 0 \qquad (j=1,\ldots,m-1)$$

Terms whose order in $\beta$ and $\tau$ is higher than the first are not entered in the parentheses (...). Since the rank of matrix $B$ is $m-1$, without loss of generality we can assume that a system of linear homogeneous equations in $w_1, \ldots, w_m$ can be solved for $w_1, \ldots, w_{m-1}$.

Therefore, for system (1) to have a periodic solution in this case, it is sufficient for the condition

$$w_m(\beta_1, \ldots, \beta_{m-1}, \tau) = 0$$

to be satisfied.

This equation can always be satisfied by a choice $\tau(\beta_1, \ldots, \beta_{m-1})$, $\tau(0, \ldots, 0) = 0$. Substituting $\tau$ in (9), we get a family of periodic solutions depending on the parameters $\beta_1, \ldots, \beta_{m-1}$. The period of this solution will be $2\pi/\lambda\ (1 + \tau(\beta_1, \ldots, \beta_{m-1})$. This proves the first part of the assertion of Theorem 1.

The proof of the second part of the theorem presents no difficulties, and is a consequence of the general assumption of the Liapunov stability theory as to a special case of the existence of a parametric solution in critical cases.

When the number of first analytic integrals is less than $m - 1$, in order that the system (1) shall have periodic solutions, some additional conditions must be satisfied. For example, the following theorem holds good:

*Theorem* 2. Let the number of integrals of type (5) be $l < m - 1$. Then system (1) will have a family of periodic solutions depending on $l$ independent parameters, provided that $|b_{ij}| \neq 0$ $(i, j = 1, \ldots, l)$, and provided that the system of equations

$$w_j(\beta_1, \ldots, \beta_{m-1}, \tau) = 0 \qquad (j=l+1, \ldots, m)$$

can be solved for $m - l - 1$ constant $\beta$'s and $\tau$'s, finding them as functions of the remaining $l$ independent parameters $\beta$.

For instance, if $\beta_1^0, \ldots, \beta_{m-1}^0, \tau^0$ is a solution of the equations $w_j(\beta_1, \ldots, \beta_{m-1}, \tau) = 0$ $(j = l + 1, \ldots, m)$ and if at the point $\beta = \beta_1^0, \ldots, \beta_{m-1} = \beta_{m-1}^0, \tau = \tau^0$ the condition

$$\left.\frac{\partial(w_{l+1}, \ldots, w_{m-1}, w_m)}{\partial(\beta_{l+1}, \ldots, \beta_{m+1}, \tau}\right|_{\beta=\beta^\circ, \tau=\tau^\circ} \neq 0$$

is satisfied, then system (1) has a periodic solution depending on $l$ parameters.

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